Complex number 3

1. Given that $z+\frac{1}{z}=1$, find the values of **(a)** $z^4+\frac{1}{z^4}$ **(b)** $z^5+\frac{1}{z^5}$.

$z+\frac{1}{z}=1 \Rightarrow z^2-z+1=0 \Rightarrow z=\frac{1\pm\sqrt{3}\,i}{2} \Rightarrow z=\cos\frac{\pi}{3}\pm i\sin\frac{\pi}{3}$(a

(a) $z^4+\frac{1}{z^4}=\left(\cos\frac{\pi}{3}\pm I\sin\frac{\pi}{3}\right)^4+\frac{1}{\left(\cos\frac{\pi}{3}\pm I\sin\frac{\pi}{3}\right)^4}=\left(\cos\frac{4\pi}{3}\pm I\sin\frac{4\pi}{3}\right)+\left(\cos\left(-\frac{4\pi}{3}\right)\pm I\sin\left(-\frac{4\pi}{3}\right)\right)$

$=\left(\cos\frac{4\pi}{3}\pm I\sin\frac{4\pi}{3}\right)+\left(\cos\frac{4\pi}{3}\mp I\sin\frac{4\pi}{3}\right)= 2\cos\frac{4\pi}{3}=-1$

(b) $z^5+\frac{1}{z^5}=\left(\cos\frac{5\pi}{3}\pm i\sin\frac{5\pi}{3}\right)+\left(\cos\frac{5\pi}{3}\mp i\sin\frac{5\pi}{3}\right)= 2\cos\frac{5\pi}{3}=1$

2. **(a)** Using deMoivre's Theorem to show that $\sin 5\theta = a\,sin^5\theta + bcos^2\theta sin^3\theta + cos^4\theta\, sin\,\theta$, where a, b and c are integers to be determined.

(b) Express $\frac{\sin 5\theta}{sin\,\theta}$ in terms of $\cos\theta$, where θ is not a multiple of π.

Hence, find the roots of the equation $16x^4-12x^2+1=0$ in trigonometric form.

(a) $\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5 = (c+i\,s)^5$, where $c=\cos\theta, s=\sin\theta$

$= c^5+5c^4(i\,s)+10\,c^3\,(i\,s)^2+10\,c^2\,(i\,s)^3+5\,c\,(i\,s)^4+(i\,s)^5$

$= (c^5-10\,c^3s^2+5\,c\,s^4)+i(s^5-10\,c^2\,s^3+5\,c^4s)$

Compare imaginary part, we have

$\sin 5\theta = s^5-10\,c^2\,s^3+5\,c^4s = sin^5\theta - 10\,cos^2\theta sin^3\theta + 5\,cos^4\theta\,sin\,\theta$

(b) $\frac{\sin 5\theta}{sin\,\theta} = sin^4\theta - 10\,cos^2\theta sin^2\theta + 5\,cos^4\theta$

$=(1-cos^2\theta)^2-10\,cos^2\theta(1-cos^2\theta)+5\,cos^4\theta$

$= (1-2cos^2\theta+cos^4\theta)-10\,cos^2\theta+10\,cos^4\theta+5\,cos^4\theta$

$= 16\,cos^4\theta-12cos^2\theta+1$

Let $x=\cos\theta$, $16x^4-12x^2+1=0 \Rightarrow 16\,cos^4\theta-12cos^2\theta+1=0$

$\frac{\sin 5\theta}{sin\,\theta}=0 \Rightarrow \sin 5\theta = 0$, where $\theta \neq k\pi$, where $k\epsilon\mathbb{Z}$.

$\theta=\frac{k\pi}{5}$, where $k\epsilon\mathbb{Z}$.

$\therefore x=\cos\frac{k\pi}{5}$, where k = 0, 1,2,3,4.

Since $x=\cos 0 = 1$ is not a root of $16x^4-12x^2+1=0$, $\therefore x=\cos\frac{k\pi}{5}$, where k = 1,2,3,4.

3. **(a)** Find the roots of $z^5 = 1$.

(b) Show that one of the roots in **(a)** is $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4} + i\frac{\sqrt{10+2\sqrt{5}}}{4}$

(c) Show that **(i)** $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$,

(ii) $|1 + \omega^2 + \omega^4| = \sqrt{\frac{3-\sqrt{5}}{2}}$.

(a) By de Moirvres' Theorem,

$z^5 = 1 \Rightarrow z = (\text{cis } 2k\pi)^{\frac{1}{5}} = \text{cis } \frac{2k\pi}{5}$, k = 0, 1, 2, 3, 4.

(b) In **(a)**, $k = 1$, $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

Put $\theta = \frac{2\pi}{5}$, then $5\theta = 2\pi$, $3\theta = 2\pi - 2\theta$, $\cos 3\theta = \cos(2\pi - 2\theta) = \cos 2\theta$

Put $x = \cos\frac{2\pi}{5}$, $4x^3 - 3x = 2x^2 - 1$, $4x^3 - 2x^2 - 3x + 1 = 0$.

Since $x \neq 1$, dividing the left-hand side of the cubic equation by $x - 1$, we get

$4x^2 + 2x - 1 = 0$, $x = \frac{-\sqrt{5}-1}{4}$ or $\frac{\sqrt{5}-1}{4}$

Rejecting the negative root, we have $x = \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$.

Using Pythagoras theorem, $\sin\frac{2\pi}{5} = \sqrt{1 - \left(\frac{\sqrt{5}-1}{4}\right)^2} = \frac{\sqrt{10+2\sqrt{5}}}{4}$

Hence $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4} + i\frac{\sqrt{10+2\sqrt{5}}}{4}$

(c) **(i)** **Method 1**

$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 1 + \text{cis}\frac{2\pi}{5} + \left(\text{cis}\frac{2\pi}{5}\right)^2 + \left(\text{cis}\frac{2\pi}{5}\right)^3 + \left(\text{cis}\frac{2\pi}{5}\right)^4$

$= 1 + \text{cis}\frac{2\pi}{5} + \text{cis}\frac{4\pi}{5} + \text{cis}\frac{6\pi}{5} + \text{cis}\frac{8\pi}{5}$, by de Moirvres' Theorem

$= 1 + \text{cis}\frac{2\pi}{5} + \text{cis}\frac{4\pi}{5} + \text{cis}\left(-\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right)$

$= 1 + \left[\text{cis}\frac{2\pi}{5} + \text{cis}\left(-\frac{2\pi}{5}\right)\right] + \left[\text{cis}\frac{4\pi}{5} + \text{cis}\left(-\frac{4\pi}{5}\right)\right]$

$= 1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 1 + 2\cos\frac{2\pi}{5} + 2\left(2\cos^2\frac{2\pi}{5} - 1\right)$

$= 1 + 2\left(\frac{\sqrt{5}-1}{4}\right) + 2\left[2\left(\frac{\sqrt{5}-1}{4}\right)^2 - 1\right] = 0$

Method 2

$$1+\omega+\omega^2+\omega^3+\omega^4 = 1+\text{cis}\,\frac{2\pi}{5}+\left(\text{cis}\,\frac{2\pi}{5}\right)^2+\left(\text{cis}\,\frac{2\pi}{5}\right)^3+\left(\text{cis}\,\frac{2\pi}{5}\right)^4$$

$$=\frac{1-\left(\text{cis}\,\frac{2\pi}{5}\right)^5}{1-\text{cis}\,\frac{2\pi}{5}} \quad \text{(Geometric series)}$$

$$=\frac{1-(\text{cis}\,2\pi)^5}{1-\text{cis}\,\frac{2\pi}{5}} \quad \text{, by de Moirvres' Theorem}$$

$$=\frac{1-(1)^5}{1-\text{cis}\,\frac{2\pi}{5}} = \mathbf{0}$$

Method 3

Since $\omega=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\neq 1$ is a root of $z^5=1$, $\omega^5=1$.

$$\therefore 1+\omega+\omega^2+\omega^3+\omega^4=\frac{1-\omega^5}{1-\omega}=\frac{1-1}{1-\omega}=\mathbf{0}$$

(c) (ii) $|1+\omega^2+\omega^4| = \left|\frac{1-(\omega^2)^3}{1-\omega^2}\right| = \left|\frac{1-\omega^6}{1-\omega^2}\right| = \left|\frac{1-\omega\omega^5}{1-\omega^2}\right| = \left|\frac{1-\omega}{1-\omega^2}\right| = \left|\frac{1}{1+\omega}\right| = \frac{1}{|1+\omega|}$

$$=\frac{1}{\left|1+\frac{\sqrt{5}-1}{4}+i\frac{\sqrt{10+2\sqrt{5}}}{4}\right|}=\frac{1}{\sqrt{\left(1+\frac{\sqrt{5}-1}{4}\right)^2+\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2}}=\frac{1}{\sqrt{\frac{\sqrt{5}+3}{2}}}=\sqrt{\frac{2}{\sqrt{5}+3}}$$

$$=\sqrt{\frac{3-\sqrt{5}}{2}}\approx \mathbf{0.6180339887499}$$

4. Show that $-\sqrt{2}+i\sqrt{2}$ is a root if $x^4+16=0$. The root $-\sqrt{2}+i\sqrt{2}$ is located on a circle of radius 2 in an Argand diagram and plot all the roots.

Method 1

Since complex roots occur in pairs, we consider the polynomial:

$$[x-(-\sqrt{2}+i\sqrt{2})][x-(-\sqrt{2}-i\sqrt{2})] = (x+\sqrt{2})^2-(i\sqrt{2})^2 = x^2+2\sqrt{2}\,x+4$$

Since irrational roots occur in pairs, we consider the polynomial:

$$[x^2+2\sqrt{2}\,x+4][x^2-2\sqrt{2}\,x+4] = (x^2+4)^2-(2\sqrt{2}\,x)^2 = x^4+16$$

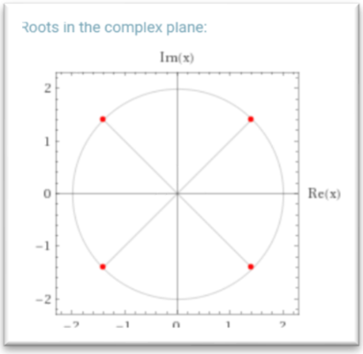
Also, note that, $x^2-2\sqrt{2}\,x+4=(x-\sqrt{2})^2-(i\sqrt{2})^2=[x-(\sqrt{2}+i\sqrt{2})][x-(\sqrt{2}-i\sqrt{2})]$

So, we get : $x^4+16=0 \Longrightarrow [x^2+2\sqrt{2}\,x+4][x^2-2\sqrt{2}\,x+4]=0$

$\Longrightarrow [x-(-\sqrt{2}+i\sqrt{2})][x-(-\sqrt{2}-i\sqrt{2})][x-(\sqrt{2}+i\sqrt{2})][x-(\sqrt{2}-i\sqrt{2})]=0$

Roots are: $x = \sqrt{2} \pm i\sqrt{2}, -\sqrt{2} \pm i\sqrt{2} = 2\left(\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i\right), 2\left(-\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i\right)$

$$x = 2\left[\cos\left(\pm\frac{\pi}{4}\right) + i\sin\left(\pm\frac{\pi}{4}\right)\right], 2\left[\cos\left(\pm\frac{3\pi}{4}\right) + i\sin\left(\pm\frac{3\pi}{4}\right)\right] \text{ (in polar form)}$$



Method 2

$x^4 + 16 = 0 \Longrightarrow x^4 = -16 = 16(\cos\pi + i\sin\pi) \Longrightarrow x = 2[\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{\frac{1}{4}}$

$x = 2\left[\cos\left(\frac{2k\pi+\pi}{4}\right) + i\sin\left(\frac{2k\pi+\pi}{4}\right)\right], k = 0,1,2,3$

$x = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right], 2\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right],$

$2\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right], 2\left[\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right].$

Note that : $2\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right] = 2\left[-\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] = \sqrt{2}(-1 + i) = -\sqrt{2} + i\sqrt{2}$ is a root of $x^4 + 16 = 0$ which is what we want in the first part of the question.

Also, other roots can be written in rectangular forms easily.

5. Solve the equation $5z^4 - z^3 + 4z^2 - z + 5 = 0$.

Since the given equation is symmetric, divide $5z^4 - z^3 + 4z^2 - z + 5 = 0$ by z^2, we get

$$5\left(z^2 + \frac{1}{z^2}\right) - \left(z + \frac{1}{z}\right) + 4 = 0 \quad \dots (1)$$

Put $\omega = z + \frac{1}{z}$, $\omega^2 = z^2 + \frac{1}{z^2} + 2$, eq (1) becomes

$5(\omega^2 - 2) - \omega + 4 = 0 \Longrightarrow 5\omega^2 - \omega - 6 = 0 \Longrightarrow \omega = -1 \text{ or } \omega = \frac{6}{5}$

When $\omega = -1$, $z + \frac{1}{z} = 1 \Longrightarrow z^2 - z + 1 = 0 \Longrightarrow z = \frac{1 \pm \sqrt{3}\,i}{2}$

When $\omega = \frac{6}{5}$, $z + \frac{1}{z} = \frac{6}{5} \Longrightarrow 5z^2 - 6z + 5 = 0 \Longrightarrow z = \frac{3 \pm 4\,i}{5}$

6. ABCD is a square with the letters in the anticlockwise order. The points A and B represents $2+3i$ and $6+i$ respectively. Find the complex number represented by C and D.

Let $z_A = 2+3i,\ z_B = 6+i.$
$z_B - z_A = (6+i) - (2+3i) = 4-2i$

Rotate $z_A z_B$ anticlockwisely by $\frac{\pi}{2}$ you get $z_A z_D$: $z_D - z_A = (z_B - z_A)e^{i\left(\frac{\pi}{2}\right)}$

$z_D - (2+3i) = (4-2i)\left[\cos\left(\frac{\pi}{2}\right) + i\,sin\left(\frac{\pi}{2}\right)\right] = (4-2i)[i] = 2+4i$

$\therefore z_D = (2+4i) + (2+3i) = \mathbf{4+7}\boldsymbol{i}$
$z_A - z_B = -4+2i$

Rotate $z_B z_A$ clockwisely by $\frac{\pi}{2}$, you get you get $z_B z_C$: $z_C - z_B = (z_B - z_A)e^{i\left(-\frac{\pi}{2}\right)}$

$z_C - (6+i) = (-4+2i)\left[\cos\left(\frac{\pi}{2}\right) - i\,sin\left(\frac{\pi}{2}\right)\right] = (-4+2i)[-i] = 2+4i$

$\therefore z_C = (2+4i) + (6+i) = \mathbf{8+5}\boldsymbol{i}$

7. The equation $z^4 - 2z^3 + kz^2 - 18z + 45 = 0$ has imaginary roots. Obtain all the roots of the equation and the value of the real constant k.

Since the given has an imaginary root, $z = \pm ai, a > 0$ are roots.
Hence $(z-ai)(z+ai) = z^2 + a^2$ is a factor of $z^4 - 2z^3 + kz^2 - 18z + 45$
$z^4 - 2z^3 + kz^2 - 18z + 45 = (z^2 + a^2)(z^2 + bz + c)$
Compare coefficients z^3 term, $b = -2$
Compare coefficients z term, $-18 = a^2 b \Longrightarrow -18 = -2a^2 \Longrightarrow a = 3$ (take $a > 0$)
Compare constant term, $45 = a^2 c = 9c \Longrightarrow c = 5$
Compare coefficients z^2 term, $k = a^2 + b = 3^2 - 2 = \mathbf{7}$
Hence the equation becomes $z^4 - 2z^3 + 7z^2 - 18z + 45 = (z^2+9)(z^2-2z+5) = 0$
The roots are $\pm\mathbf{3}\boldsymbol{i}, \mathbf{1} \pm \mathbf{2}\boldsymbol{i}$.

8. **(a)** Let $z = -2-3i$, find z^2.
(b) Hence solve the equations : (i) $w^2 + 4w = -9+12i$
(ii) $w^4 + 4w^2 = -9+12i.$

(a) $z^2 = -5+12i$
(b)(i) $w^2 + 4w = -9+12i \Longrightarrow w^2 + 4w + 4 = -5+12i \Longrightarrow (w+2)^2 = (-2-3i)^2$
$(w+2)^2 - (-2-3i)^2 = 0$

$[(w+2)+(-2-3i)][(w+2)-(-2-3i)]=0$
$[w^2-3i][w^2+4+3i]=0$
$w=3i$ or $w=-4-3i$

(ii) $w^4+4w^2=-9+12i \Longrightarrow w^4+4w^2+4=-5+12i \Longrightarrow (w^2+2)^2=(-2-3i)^2$
$(w^2+2)^2-(-2-3i)^2=0$
$[(w^2+2)+(-2-3i)][(w^2+2)-(-2-3i)]=0$
$[w^2-3i][w^2+4+3i]=0$
$w^2=3i$ or $w^2=-3i-4$

$w^2=3\,cis\,\frac{\pi}{2}$ or $w^2=5\,cis(-2.489)$

$w=\sqrt{3}\left[cis\left(2k\pi+\frac{\pi}{2}\right)\right]^{\frac{1}{2}}$ or $\sqrt{5}\,[cis\,(2k\pi-2.489)]^{\frac{1}{2}}$

$=\sqrt{3}\left[cis\left(\frac{2k\pi+\frac{\pi}{2}}{2}\right)\right]$ or $\sqrt{3}\left[cis\left(\frac{2k\pi-2.489}{2}\right)\right]$, k = 0, 1.

$w_1=\sqrt{3}\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right]=\sqrt{\frac{3}{2}}(1+i)\approx 1.2247449+1.2247449i$

$w_2=\sqrt{3}\left[\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right]=-\sqrt{\frac{3}{2}}(1+i)\approx -1.2247449-1.2247449i$

$w_3=\sqrt{3}[\cos(-1.2445)+i\sin(-1.2445)]\approx 0.555186-1.64066\,i$
$w_4=\sqrt{3}[\cos\,(\pi-1.2445)+i\sin(\pi-1.2445)]\approx -0.555186\,+1.64066\,i$

9. If $z=\cos\theta+i\sin\theta$, show that $\frac{1}{1+z^2}=\frac{1}{2}(1-i\,tan\,\theta)$ and write $\frac{1}{1-z^2}$ in similar form.

$z^2=(\cos\theta+i\sin\theta)^2=\cos 2\theta+i\sin 2\theta$

$1+z^2=(1+\cos 2\theta)+i\sin 2\theta=2cos^2\theta+2i\,sin\theta\cos\theta=2\cos\theta\,(\cos\theta+i\sin\theta)$

$\frac{1}{1+z^2}=\frac{1}{2\cos\theta(\cos\theta+i\sin\theta)}=\frac{1}{2\cos\theta}[\cos(-\theta)+i\sin(-\theta)]=\frac{1}{2\cos\theta}(\cos\theta-i\sin\theta)=\boldsymbol{\frac{1}{2}(1-i\,tan\,\theta)}$

$1-z^2=(1-\cos 2\theta)-i\sin 2\theta=2sin^2\theta-2i\,sin\theta\cos\theta=2\sin\theta\,(\sin\theta-i\cos\theta)$

$=2\sin\theta\left[\cos\left(\frac{\pi}{2}-\theta\right)-i\sin\left(\frac{\pi}{2}-\theta\right)\right]=2\sin\theta\left\{\cos\left[-\left(\frac{\pi}{2}-\theta\right)\right]+i\sin\left[-\left(\frac{\pi}{2}-\theta\right)\right]\right\}$

$\frac{1}{1-z^2}=\frac{1}{2\sin\theta\left\{\cos\left[-\left(\frac{\pi}{2}-\theta\right)\right]+i\sin\left[-\left(\frac{\pi}{2}-\theta\right)\right]\right\}}=\frac{1}{2\sin\theta}\left\{\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right\}=\frac{1}{2\sin\theta}(\sin\theta+i\cos\theta)$

$=\boldsymbol{\frac{1}{2}(1+i\,cot\,\theta)}$

10. (a) Let $z = 1 + i$, find $z^n, n = 1,2,3,4, \ldots$ in Cartesian form.

(b) Find z^n in polar form.

(c) Explain, without plotting, how you can represent z^n in Argand diagram.

n	
1	$1+i$
2	$2i$
3	$-2+2i$
4	-4
5	$-4-4i$
6	$-8i$
7	$8-8i$
8	16
9	$16+16i$
10	$32i$
11	$-32+32i$
12	-64
13	$-64-64i$
14	$-128i$
15	$128-128i$

(a) $z^n = \begin{cases} (-4)^k & n = 4k \\ (-2)^{2k-1}(1-i) & n = 4k-1 \\ (-1)^{k-1}2^{2k-1}i & n = 4k-2 \\ (-4)^{k-1}(1+i) & n = 4k-3 \end{cases}$ where $k = 1,2,3 \ldots.$

(b) $z = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$\Longrightarrow z^n = \left(\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right), n = 1,2,3,4, \ldots$

(c) It is easier to write the answers in polar form in (b) and the Argand diagram is easier to draw.
If we put $z^0 = (1+i)^0 = 1$

$z^1 = \sqrt{2}\, cis\, \frac{\pi}{4}$ can be plot by rotating anti-clockwisely the vector z^0 by an angle $\frac{\pi}{4}$ and lengthen the radius by a factor of $\sqrt{2}$. Similarly by rotating z^1 anti-clockwisely by an angle $\frac{\pi}{4}$ and lengthen the radius of z^1 by a factor of $\sqrt{2}$, we can get z^2,… We can get a spiral of points.

Yue Kwok Choy
24/10/2017